A DYNAMIC MODEL OF SHIRKING AND UNEMPLOYMENT:

PRIVATE SAVING, PUBLIC DEBT AND OPTIMAL TAXATION

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Abstract

This paper introduces private saving and public debt into the shirking-unemployment model of Shapiro and Stiglitz (1984), while relaxing their exclusive focus on steady states. After generalizing their no-shirking constraint to accommodate asset accumulation, and demonstrating that the resulting economy's equilibrium is saddle-path stable, we use our dynamic model to obtain significant departures from the Shapiro-Stiglitz prescriptions for optimal policy. Most notably, wage income should be taxed (not subsidized) in the long run if the labor market is sufficiently distorted. Furthermore, interest income should be (exhaustively) taxed only during an initial interval of time, as in Chamley's (1986) full-employment model.

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1. Introduction

A prominent explanation of unemployment is based on the efficiency-wage hypothesis, according to which labor productivity is positively related to the real wage rate. Shapiro and Stiglitz (1984) use an optimizing framework to develop an important version of this explanation, in which firms set the wage rate to prevent shirking, because they cannot perfectly monitor labor effort. The no-shirking wage is greater than the marginal cost of effort, and thus the efficiency-wage equilibrium produces a labor-market distortion, characterized by involuntary unemployment. This unemployment serves as a "worker discipline device", because employees are fired if caught shirking.[1]

Two significant limitations of the Shapiro-Stiglitz (1984) model are its exclusive focus on steady states, and its exclusion of the saving/investment process. Kimball (1994) and Phelps (1994, chap. 15) extend the model to relax, respectively, the first and second of these limitations.

However, as Phelps (1994, p.264) points out, "The intertemporal microeconomics of saving and shirking in the presence of imperfect monitoring is … not an analytically convenient subject." The main challenge is that the no-shirking wage depends on wealth, which varies across individuals because of their different employment (unemployment) histories. Thus, the distribution of wealth needs to be pinned down, in order to derive a tractable relation for determining the no-shirking wage. Brecher, Chen and Choudhri (2002) make some progress in this regard, but only for steady states.

The present paper generalizes the Shapiro-Stiglitz (1984) model to incorporate both optimal saving and transitional dynamics. To simplify the analysis, we assume that each

[1] For an alternative type of disciplinary mechanism, see Alexopoulos (2003, 2006), who assumes that detected shirkers are fined (a fraction of their wage) rather than fired.

household makes saving and shirking decisions for its many members, using an aggregate (household) utility function. This simplification suppresses the role of wealth differences across individuals, and enables us to derive a no-shirking wage relation for the entire transitional path (including steady state), thereby nesting the Shapiro-Stiglitz (1984) analysis as a special case.

Our model also provides a representative-agent framework for analyzing optimal taxes (subsidies) on capital and labor income in the presence of shirking and unemployment. Intuitively, it seems reasonable to conjecture that the government should continuously implement a wage subsidy financed by an interest tax. Indeed, Shapiro and Stiglitz (1984) show that the optimal policy prescription in their model—with its fixed stock of capital—is to tax away all interest income and use this revenue to subsidize wages.

Our analysis, however, qualifies this Shapiro-Stiglitz (1984) prescription in three significant ways. First, although the tax on capital income should indeed be maximal in the short run, this tax should be completely eliminated in the long run. Second, rather than immediately spend all short-run interest-tax revenue on labor subsidies, it might be better for the government to divert some of this revenue toward accumulation of public assets (negative debt) that finance a wage subsidy in the long run. Third, and more surprisingly, it might even be optimal to tax (rather than subsidize) labor in steady state.

In models of full employment, Chamley (1980, 1986) and Judd (1985) show that the optimal tax on capital income is zero in the long run, if the economy converges to steady state. However, this result need not hold in the presence of labor-market distortions, as Domeij (2005) and Arseneau and Chugh (2006) demonstrate in search-and-matching models of frictional unemployment. By restoring the Chamley-Judd result, our shirking-unemployment analysis indicates that a zero (long-run) tax on capital income may remain optimal even when the labor market is distorted.

Under optimal taxation, steady-state equilibrium depends on the initial levels of aggregate capital, total employment and public debt. To explore this link, we use numerical methods to solve our model dynamically, for alternative sets of initial conditions. This procedure allows us to construct and analyze concrete examples with a positive tax on wages in steady state.

In sum, the present paper makes several distinct contributions. First, it develops a tractable dynamic model of efficiency-wage unemployment with optimal saving, and explores both transitional and steady-state properties of this model. Then, to demonstrate an application of the model, we derive the dynamic counterpart to the policy prescription of Shapiro and Stiglitz (1984). This application also suggests that optimal fiscal policy depends importantly on whether unemployment is of the efficiency-wage variety versus some other (e.g., frictional) type. Finally, using a numerical solution of the present model, we construct examples in which optimal policy departs significantly from the Shapiro-Stiglitz prescription.

Sections 2 and 3, respectively, derive the fundamental no-shirking condition and the related equilibrium wage. To elucidate the basic workings of the model, section 4 investigates the steady-state equilibrium and dynamic stability of our efficiency-wage economy in the laissez-faire case, free from government intervention. Then, section 5 tackles the more complex problem of optimal taxation by an active government. To shed additional light on this problem, section 6 undertakes numerical analysis of our model. We offer some concluding remarks in section 7.

2. To Shirk or Not to Shirk

This section discusses optimization by households, and shows that there will be no shirking if and only if a certain condition is satisfied. For this purpose, we set up a one-good model that incorporates key features of the Shapiro-Stiglitz (1984) efficiency-wage framework, but allows also for asset accumulation and transitional dynamics. At time t, the employment status of individual i is represented by $z_i(t)$, which equals 1 or 0 as the individual is respectively employed or unemployed. The shirking behavior of an employee is given by $S_i(t)$, which (for simplicity) equals 1 if there is shirking or 0 otherwise. Thus, the individual supplies a quantity of effort equal to $z_i(t)[1-S_i(t)]$. Letting $C_i(t)$ denote the amount consumed at time t, assume that the utility function of an individual takes the following standard form:

$$U_i = \int_0^\infty e^{-\rho t}\{C_i^{1-\theta}/(1-\theta) - l[z_i(1-S_i)]\}dt, \qquad (1)$$

where ρ is the constant rate of time preference; $\theta > 0$; l is an increasing function; and the time argument is suppressed henceforth, unless needed for clarity.

For non-shirking employees, the fixed probability of being exogenously separated from a job is b per unit time. Shirkers face a higher probability $b+q$ of job loss, where q represents the fixed probability per unit time of being detected shirking. Unemployed workers, on the other hand, have a probability $a(t)$ per unit time of acquiring a job. To abstract from complications due to wealth differences among agents, assume that the economy has many identical households, each consisting of numerous individuals. Each household chooses the same consumption level and shirking behavior for all of its members, to maximize a utility function that is an average of the utilities of individual members.

More specifically, assume a continuum of households in the unit interval, and similarly let each household have a continuum of individuals indexed by $i \in [0,1]$. The utility function of a typical household is then given by $U = \int_0^1 U_i di$. Using this function and (1), while setting $C_i = C$ and $S_i = S$, obtain

$$U = \int_0^\infty e^{-\rho t}[C^{1-\theta}/(1-\theta) - \delta Z(1-S)]dt, \tag{2}$$

where $\delta \equiv l(1) - l(0)$; $l(0) = 0$ by normalization; and $Z \equiv \int_0^1 z_i di$, which equals the proportion of members who are employed.

Let $w(t)$ and $r(t)$ represent the pre-tax wage and interest rates. The government levies proportional taxes on wage and interest incomes at the rates $\tau_w(t) \le 1$ and $\tau_r(t) \le 1$, respectively. (If a tax rate is negative, it represents a positive subsidy.) Households choose C and S to maximize U in (2), subject to the following constraints:

$$\dot{X} = \tilde{r}X + \tilde{w}Z - C, \tag{3}$$

$$\dot{Z} = a(1-Z) - (b+qS)Z, \tag{4}$$

$$S(1-S) = 0, \tag{5}$$

$$X(0) = X_0, \; Z(0) = Z_0, \tag{6}$$

where X is household wealth, while $\tilde{w} \equiv w(1-\tau_w)$ and $\tilde{r} \equiv r(1-\tau_r)$ are the take-home wage and interest rates. Constraints (3) and (4) represent wealth accumulation and employment dynamics; (5) restricts the choice of S to its two possible values (0 and 1); and (6) gives the initial conditions.

The Lagrangean function for the household is

$$L^h = C^{1-\theta}/(1-\theta) - \delta Z(1-S) + \mu(\tilde{r}X + \tilde{w}Z - C)$$
$$+ m[a(1-Z) - (b+qS)Z] + \lambda S(1-S)\,, \tag{7}$$

where $L^h - \lambda S(1-S)$ represents the current-value Hamiltonian; μ and m are co-state variables, which can be interpreted as shadow values of wealth and employment, respectively; and λ is a Lagrange multiplier. The necessary conditions for the household's maximization problem are

$$\partial L^h / \partial C = C^{-\theta} - \mu = 0\,, \tag{8}$$

$$\partial L^h / \partial S = (\delta - mq)Z + \lambda(1-2S) = 0\,, \tag{9}$$

$$\dot{\mu} = \rho\mu - \partial L^h / \partial X = \mu(\rho - \tilde{r})\,, \tag{10}$$

$$\dot{m} = \rho m - \partial L^h / \partial Z = (\rho + a + b + qS)m + \delta(1-S) - \tilde{w}\mu\,, \tag{11}$$

$$\lim_{t\to\infty} \mu X e^{-\rho t} = \lim_{t\to\infty} mZe^{-\rho t} = 0\,, \tag{12}$$

as well as constraints (3) – (6).[2]

The second-order condition with respect to S is $\partial^2 L^h / \partial S^2 = -2\lambda \le 0$, which implies that $\lambda \ge 0$. Clearly, $\lambda > 0$ to satisfy (9) if $\delta \ne mq$ (and $Z > 0$), in which case S equals 0 or 1 as δ is respectively less or greater than mq. On the other hand, if $\delta = mq$, $\lambda = 0$ to satisfy (9), which implies that S equals either 0 or 1 (i.e., households are

[2] Throughout this paper, in stating transversality conditions like (12), we follow the general approach taken by Barro and Sala-i-Martin (2004, p. 615) for the infinite-horizon case.

indifferent between shirking and not shirking). In this case of indifference, adopt the Shapiro-Stiglitz (1984) convention that households choose $S = 0$. To summarize, we have the following result.

Proposition 1. There is no shirking at time t if and only if

$$m(t) \geq \delta / q , \qquad (13)$$

where the equality implies indifference between shirking and not shirking, while the inequality indicates a definite preference for not shirking.

To interpret this no-shirking condition, note first that δ represents the household's gain from shirking, by which effort is being withdrawn. Since m is the shadow value of employment, moreover, qm is the expected loss from shirking that leads to firing if detected. Condition (13) thus simply requires that the gain (δ) from shirking not exceed the expected loss (qm) from this behavior.

3. No-Shirking Wage

Assuming a large number of identical firms, we now derive a formula for the equilibrium wage at each point in time. In accordance with the conventional story, the forces of competition drive the market wage to the level that keeps households indifferent between shirking and not shirking.[3] Thus, by proposition 1,

$$m = \delta / q \qquad \text{for all } t, \qquad (14)$$

in competitive equilibrium.

If the market wage were such that $m > \delta / q$ throughout an interval, each firm would be tempted to reduce its wage slightly in the interval. This reduction would not

[3] For a lucid account of this process, see Phelps (1994, chap. 1).

cause the firm's employees to shirk, given a market wage better than needed to prevent shirking. (They would not quit for higher-paying jobs, because separation from the firm entails an immediate spell of unemployment.) Therefore, any market wage inconsistent with (14) would be unsustainable.[4]

To derive the equilibrium wage, note that (14) implies $\dot{m} = 0$, substitute these two equalities into (11), set $S = 0$, and obtain the following result.

Proposition 2. At each point in time, the take-home wage is given by

$$\tilde{w} = (\rho + a + b + q)\delta / q\mu \,, \tag{15}$$

which represents the no-shirking wage that leaves households indifferent between shirking and not shirking.

In light of this result, set $S = 0$ henceforth.

Equation (15) simplifies to the no-shirking wage of Shapiro and Stiglitz (1984) in the following special case: $\theta = 0$, given that their utility function is linear in consumption; and $\dot{X} \equiv 0$, because they do not allow for saving. (With this last restriction, μ becomes a Lagrange multiplier instead of a co-state variable.) Under these assumptions, the equilibrium wage is still given by (15), but with $\mu \equiv 1$.

It is interesting to observe that X does not directly enter (15), even though the Phelps (1994, chap. 15) and Brecher-Chen-Choudhri (2002) analyses suggest that the no-shirking wage depends on wealth. Evidently, this wealth effect in the present model is fully captured by the presence of μ (the shadow value of X) in (15).

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[4] To formalize this heuristic argument, we could extend the model to let each household's shirking decision vary across firms if they offer different wage rates. Such an extension would complicate the exposition, without affecting our results.

Aggregate output is given by $F(K,Z)$; where F is a neoclassical production function that exhibits homogeneity of degree one, strict quasi-concavity and the Inada conditions; K represents the total stock of (fully utilized) capital; and, under our continuum-of-households assumption above, the aggregate level of employment equals Z. Output can be consumed or added to the capital stock, which is consumable but not otherwise subject to depreciation. Profit maximization yields the usual marginal-productivity conditions, which are

$$r = \partial F(K,Z)/\partial K = f'(K/Z), \tag{16}$$

$$w = \partial F(K,Z)/\partial Z = f(K/Z) - (K/Z)f'(K/Z), \tag{17}$$

where $f(K/Z) \equiv F(K/Z,1) = F(K,Z)/Z$.

When the economy is in equilibrium, we can use (15) to solve for a endogenously, and substitute this solution into (4) to obtain

$$\dot{Z} = (\tilde{w}\mu q/\delta - \rho - b - q)(1-Z) - bZ, \tag{18}$$

after remembering that $S = 0$. Constraint (18) shows how the equilibrium wage affects the path of employment over time. In section 5 below, some of our optimal-tax results are explained by the facts that X and $\tilde{r}$ do not enter (18), while this constraint includes $\tilde{w}$ and μ as (and only as) the product $\tilde{w}\mu$.

4. Laissez-Faire Economy

For the simple case without government intervention, this section derives the steady-state equilibrium of our efficiency-wage economy, and shows that the dynamic system of the model is saddle-path stable. In this laissez-faire case, $\tau_r \equiv \tau_w \equiv 0$, $\tilde{r} \equiv r$, $\tilde{w} \equiv w$ and $X \equiv K$.

Thus, given (16) and (17), rewrite (3), (18) and (10) as

$$\dot{K} = Zf(K/Z) - \mu^{-1/\theta}, \tag{19}$$

$$\dot{Z} = \{[f(K/Z) - (K/Z)f'(K/Z)]\mu q/\delta - \rho - b - q\}(1-Z) - bZ, \tag{20}$$

$$\dot{\mu} = \mu[\rho - f'(K/Z)], \tag{21}$$

using (8) to substitute for C (and recalling that $S = 0$). Equations (19) – (21) describe the dynamic system for K, Z and μ.

To find the steady-state equilibrium, set $\dot{K} = \dot{Z} = \dot{\mu} = 0$. Then, from (21), solve for $K/Z = \bar{k}$, where overbars indicate steady-state values. Next, substitute this solution into (19) and (20) to solve simultaneously for $\mu = \bar{\mu}$ and $Z = \bar{Z}$. Thus, $K = \bar{K} \equiv \bar{k}\bar{Z}$

To show that the dynamic system is saddle-path stable, linearize (19) – (21) in the neighborhood of steady state, using a Taylor approximation. The resulting set of equations is

$$\begin{pmatrix} \dot{K} \\ \dot{Z} \\ \dot{\mu} \end{pmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{pmatrix} K - \bar{K} \\ Z - \bar{Z} \\ \mu - \bar{\mu} \end{pmatrix}, \tag{22}$$

where $p_{11} = \rho$, $p_{12} = \bar{w}$, $p_{13} = \theta^{-1}\bar{\mu}^{-1/\theta-1}$, $p_{21} = -\bar{k}f''(\bar{k})\bar{\mu}q(1-\bar{Z})/(\bar{Z}\delta)$, $p_{22} = -\bar{k}p_{21} - \bar{a} - b$, $p_{23} = (1-\bar{Z})\bar{w}q/\delta$, $p_{31} = -\bar{\mu}f''(\bar{k})/\bar{Z}$, $p_{32} = -\bar{k}p_{31}$, and $p_{33} = 0$. The expressions for p_{11} and p_{33} follow from the fact that $\bar{r} = \rho$ by (21). To sign p_{21} and p_{31}, note that $f''(\bar{k}) < 0$ by diminishing marginal productivity.

Since the sum of the eigenvalues of a square matrix equals the trace of the matrix,

$$E_1 + E_2 + E_3 = \rho - \bar{a} - b + f''(\bar{k})\bar{k}^2\bar{\mu}q(1-\bar{Z})/(\bar{Z}\delta), \tag{23}$$

where E_1, E_2 and E_3 denote the eigenvalues of the (square) matrix in (22). Because the product of the eigenvalues of a square matrix equals the determinant of the matrix,

$$E_1 E_2 E_3 = -f''(\bar{k})\bar{\mu}[\bar{w}f(\bar{k})(1-\bar{Z})q/\delta + (\bar{a}+b)\theta^{-1}\bar{\mu}^{-1/\theta-1}]/\bar{Z}\ . \qquad (24)$$

If not for the presence of ρ in (23), the right-hand side of this equation would always be negative. To ensure this outcome, simply assume that $\rho \le b$, although this assumption is clearly stronger than required. In this case, (23) implies that at least one eigenvalue is negative. Then, observing that the right-hand side of (24) is always positive, we know that exactly two eigenvalues are negative.

With two negative eigenvalues, the dynamic system is saddle-path stable, and the stable arm (manifold) has two dimensions. We therefore have the following result.

Proposition 3. Starting from any arbitrary combination of K_0 *and* Z_0 *close to* $\bar{K}$ *and* $\bar{Z}$, $\mu(0)$ *takes the value needed to reach the stable arm, along which the economy converges to the steady-state equilibrium.*

5. Optimal Taxation of Capital and Labor Incomes

We now turn to the more complex case of optimal policy for an active government. Subsections 5.1 and 5.2, respectively, outline the maximization problem and characterize the social distortion. Then, subsections 5.3 and 5.4 highlight some properties of optimal intervention in the markets for capital and labor, respectively.

5.1. *Government's Problem*

Let the government tax (subsidize) wage and interest incomes optimally, by choosing $\tilde{w}$ and $\tilde{r}$. This choice is equivalent to setting τ_w and τ_r, since w and r are determined by (17) and (16), respectively. Thus, the government is constrained to set

$$\tilde{r} \geq 0, \tag{25}$$

as interest income cannot be taxed at a rate of more than 100%. A similar restriction on $\tilde{w}$ is redundant, since the no-shirking wage is positive by (15).

The government can also issue debt, which is a perfect substitute for claims on capital. With $D(t)$ denoting the stock of this debt at time t,

$$\dot{D} = \tilde{r}(D + K) + \tilde{w}Z - F(K, Z), \tag{26}$$

which represents the deficit in the government's instantaneous budget.[5] If this deficit is negative (indicating a surplus), public wealth ($-D$) is accumulating.

In addition to initial conditions (6), we have

$$K(0) = K_0,\ D(0) = X_0 - K_0, \tag{27}$$

where the second equality follows from the fact that $X \equiv D + K$. In light of this identity, (26), (19) and (8) imply (3).

The optimal-policy problem is to maximize the representative household's utility in (2), by choosing a combination of $\tilde{w}$ and $\tilde{r}$ for each t,[6] subject to the following constraints: transition equations (18), (19) and (26) for state variables Z, K and D; household-optimization conditions (8) and (10) involving μ, which the government

[5] Without significantly affecting the results below, we could easily incorporate an exogenous flow of government consumption that is not included in the utility function, but is subtracted from and added to the right-hand sides of (19) and (26), respectively.
[6] In light of the well-known time-inconsistency problem that Turnovsky (2000, p.409) addresses, make the usual assumption that the government is fully and credibly committed to honor forever the tax choices that it announces at time 0.

treats as a state variable; non-negativity restriction (25) on the interest rate; and initial conditions (6) and (27).[7] Write the Lagrangean for this problem as

$$L^g = (\mu^{-1/\theta})^{1-\theta}/(1-\theta) - \delta Z + \eta[F(K,Z) - \mu^{-1/\theta}] + \nu[(\tilde{w}\mu q/\delta - \rho - b - q)(1-Z) - bZ]$$

$$+ \pi[\tilde{r}(D+K) + \tilde{w}Z - F(K,Z)] + \gamma\mu(\rho - \tilde{r}) + \psi\tilde{r}\,, \tag{28}$$

where η, ν, π and γ are co-state variables; ψ is a Lagrange multiplier; and $L^g - \psi\tilde{r}$ is the current-value Hamiltonian.

The necessary conditions (additional to the relevant constraints) for the government's problem are

$$\partial L^g / \partial\tilde{w} = \pi Z + \nu\mu(1-Z)q/\delta = 0\,, \tag{29}$$

$$\partial L^g / \partial\tilde{r} = \pi(D+K) - \gamma\mu + \psi = 0\,,\ \psi \geq 0\,,\ \psi\tilde{r} = 0\,, \tag{30}$$

$$\dot{\eta} = \rho\eta - \partial L^g / \partial K = \eta(\rho - r) - \pi(\tilde{r} - r)\,, \tag{31}$$

$$\dot{\nu} = \rho\nu - \partial L^g / \partial Z = \nu(\tilde{w}\mu/\delta - 1)q + \delta - \eta w - \pi(\tilde{w} - w) \tag{32}$$

$$\dot{\pi} = \rho\pi - \partial L^g / \partial D = \pi(\rho - \tilde{r})\,, \tag{33}$$

$$\dot{\gamma} = \rho\gamma - \partial L^g / \partial\mu = \gamma\tilde{r} - (\eta - \mu)\mu^{-1/\theta - 1}/\theta - \nu\tilde{w}(1-Z)q/\delta\,, \tag{34}$$

$$\lim_{t\to\infty}\eta K e^{-\rho t} = \lim_{t\to\infty}\nu Z e^{-\rho t} = \lim_{t\to\infty}\pi D e^{-\rho t} = \lim_{t\to\infty}\gamma\mu e^{-\rho t} = \gamma(0) = 0\,, \tag{35}$$

where r and w satisfy (16) and (17). The last equality in transversality conditions (35) is needed because $\mu(0)$ is unconstrained.

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[7] Because we are implicitly assuming an interior solution (except for $\tilde{r}$), it is not necessary to include explicitly the following additional constraints: $Z \leq 1$; $a \geq 0$; and $X \geq 0$, to avoid the moral hazard of unsecured private borrowing.

5.2. *Social Distortion*

An important implication of (10) and (33) is that

$$\dot{\phi} = 0 \qquad \text{for all } t, \tag{36}$$

where $\phi \equiv \pi / \mu$. Since π (the shadow value of public debt) can be interpreted as the marginal excess burden of using distortionary (rather than lump-sum) taxation,[8] ϕ is a measure of this social burden in units of household wealth (whose shadow price is μ). This measure of the distortion is constant along the optimal path, according to (36). In reaching this conclusion, note the important fact that ν is absent from (33) because (as explained in section 3 above) constraint (18) does not include $X(\equiv D+K)$.

Given the state of involuntary unemployment, it is natural to focus on the case in which the initial employment level (Z_0) is less than socially optimal. Thus, assume that

$$\nu(0) > 0, \tag{37}$$

which means that a marginal rise in employment at time 0 would increase the present discounted value of lifetime utility.[9] From (37) and (29), $\pi(0) < 0$. Therefore, in light of (36),

$$\pi < 0 \qquad \text{for all } t. \tag{38}$$

Conditions (38) and (29) imply that

$$\nu > 0 \qquad \text{for all } t. \tag{39}$$

Thus, ceteris paribus, a marginal rise in employment would always increase the current value of (remaining) lifetime utility. In other words, we have the following result.

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[8] For this interpretation, see Atkinson and Stern (1974).
[9] This is analogous to the Shapiro-Stiglitz (1984) assumption that the marginal product of labor exceeds the disutility of effort in their no-savings model.

Proposition 4. At each instant in time, employment is below the level that would maximize the current value of lifetime utility (i.e., make $\nu = 0$).

Nevertheless, stimulating employment by raising $\tilde{w}$ would also increase D [via (26)], whose shadow value (π) is negative [by (38)]. For this reason, the government would refrain from expanding employment beyond the level consistent with (29).

5.3. *Interest Tax*

Despite the presence of unemployment, the optimal path of the tax on interest income is essentially the same as in Chamley's (1986) full-employment model. For example, given our last equality in (35), conditions (30) and (38) imply that $\psi(0) > 0 = \tilde{r}(0)$, which means that interest income should be taxed at a rate of 100% in some initial interval. On the other hand, interest should not be taxed in steady state (where $\dot{\mu} = \dot{\pi} = 0$) , since $\tilde{r} = r$ from (10), (31) and the fact that $\eta > \pi$.[10]

Therefore, the initial interval (with $\tau_r \equiv 1$) must eventually come to an end, followed immediately by a second interval with $\psi \equiv 0 \equiv \dot{\psi}$. To fully characterize this second interval, use the facts that $\phi \equiv \pi / \mu$ and $X \equiv D + K$, to rewrite (30) as $\psi = (\gamma - \phi X)\mu$. Then, differentiate this equation with respect to t; use (3), (10), (34) and (36) to substitute for $\dot{X}$, $\dot{\mu}$, $\dot{\gamma}$ and $\dot{\phi}$; simplify with the help of (8), (29) and the above formula for ψ ; and obtain the following pair of conditions:

$$\dot{\psi} = \rho\psi + NC/\theta , \qquad N \equiv \mu - \eta + \theta\pi , \tag{40}$$

$$\dot{N} = (\rho - \tilde{r})N + (\eta - \pi)(r - \tilde{r}) , \tag{41}$$

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[10] Recall (38), and note that η (the shadow price of capital) is normally positive. The expression $\eta - \pi$ equals the social marginal cost of government consumption (of the type mentioned in footnote 5 above), and is strictly positive, in accordance with Chamley (1980).

after using (31) and (33) to eliminate $\dot{\eta}$ and $\dot{\pi}$. Condition (40) implies that $N \equiv 0 \equiv \dot{N}$ (given $\psi \equiv 0 \equiv \dot{\psi}$), and hence (41) implies that $\tilde{r} \equiv r$ throughout the second interval (since $\eta > \pi$).

To prove (by contradiction) that the second interval (with $\tau_r \equiv 0$) lasts forever[11], suppose instead that it ends at some point in time. Then, the following conditions clearly must hold immediately afterward: $\psi > 0$ (by the above supposition), implying that $\dot{\psi} > 0$; and $\dot{N} > 0$ [from (41), given N close to zero (by continuity), and $\tilde{r} = 0$ (because $\psi > 0$)], implying that $N > 0$. Thus, by (40) and (41), $\psi > 0$ forever after— thereby contradicting the fact that $\psi = 0$ (because $\tilde{r} > 0$) in steady state.

In summary, we have the following result.

Proposition 5. There exists a time T (>0) *such that interest income should be maximally taxed (i.e.,* $\tilde{r} = 0$*) for* $t < T$*, but completely untaxed (i.e.,* $\tilde{r} = r$*) for* $t > T$.

Thus, interest should be taxed only in the short run (at 100%), even though the labor-market distortion (indicated by $\nu > 0$) persists along the optimal path in its entirety, including the long run.

To understand why optimal taxation of capital income in our shirking-unemployment model is qualitatively the same as in the standard full-employment case, it is helpful to start with Chari and Kehoe's (1999, p.1697) following observation about additional restrictions on the tax system: "Zero capital income taxation in the steady state is optimal if the extra constraints do not depend on the capital stock and is not optimal if these constraints depend on the capital stock (and, of course, are binding)". Because our

[11] Brecher and Chen (2006) identify and fill a serious gap in Chamley's (1986) own proof of this result.

additional restriction—namely, constraint (18)—does not depend (directly) on the capital stock, we find that capital income should be untaxed in steady state, even though the labor market remains distorted. Similarly, since (18) does not include $\tilde{r}$, ν is absent from (30), which allows us to determine that interest should be taxed initially at a rate of 100%. Furthermore, as can be readily verified, the rest of proposition 5 relies also on the fact that $\tilde{w}$ and μ enter (18) as (and only as) the product $\tilde{w}\mu$.

On the other hand, in search-and-matching models of frictional unemployment, Domeij (2005) and Arseneau and Chugh (2006) find that the capital-income tax should generally differ from zero in steady state. The reason for this finding is the presence of extra constraints that do depend on the capital stock.[12] As suggested by this difference between their result and ours, the reason for unemployment matters to the optimal taxation of capital income.

5.3. *Wage Tax*

The difference between short- and long-run optimality is not so clear-cut for the wage tax. In fact, it would be reasonable to conjecture that the government should use its short-run capital-tax revenue for subsidizing employment at every t, because $\nu > 0$ always. Surprisingly, however, we now show that the tax on wage income might actually be positive in steady state.

For this purpose, several features of steady-state equilibrium are used. First, set $\dot{Z} = 0$ in (18), and solve for Z as a function $\tilde{Z}(\tilde{w}\mu) \equiv 1 - b/[(\tilde{w}\mu/\delta - 1)q - \rho]$, enabling us to write

$$Z = \tilde{Z}(\tilde{w}\mu), \quad \tilde{Z}'(\tilde{w}\mu) > 0 . \qquad (42)$$

[12] See Domeij's (ii) in problem (P.3), as well as Arseneau and Chugh's (16) and (18).

Next, set $\dot{K}=0$ in (19) to obtain $(\overline{w}+\rho\bar{k})Z=\mu^{-1/\theta}$, where $\bar{k}$ and $\overline{w}$ are the same as in the laissez-faire case; use (42); and solve (implicitly) for the following functional relationship:

$$\mu=\tilde{\mu}(\tilde{w}), \qquad -\mu/\tilde{w}<\tilde{\mu}'(\tilde{w})<0. \tag{43}$$

Then, set $\dot{v}=0$ in (32), and solve for

$$v=[(\eta\overline{w}-\delta)+(\tilde{w}-\overline{w})\phi\mu]\delta/[(\tilde{w}\mu-\delta)q], \tag{44}$$

recalling that $\phi\equiv\pi/\mu$. Finally, set $N=0$ (which holds for $t>T$) in the identity of (40), to show that

$$\eta=(\phi\theta+1)\mu. \tag{45}$$

Now, rewrite (29) as $\phi=-v(1-Z)q/(\delta Z)$, and use steady-state conditions (44) and then (45) to eliminate v and then η, thereby yielding

$$\phi=(\delta-\overline{w}\mu)/\Omega, \qquad \Omega\equiv(\tilde{w}\mu-\delta)Z/(1-Z)+\theta\overline{w}\mu+(\tilde{w}-\overline{w})\mu>0. \tag{46}$$

The inequality in (46) clearly holds if $\tilde{w}>\overline{w}$, since (15) implies that $\tilde{w}\mu>\delta$. If instead $\overline{w}>\tilde{w}$, then $\overline{w}\mu>\delta$ by (15), and hence $\Omega>0$ (because $\phi<0$) as before.

Into (46), substitute steady-state conditions (42) and (43), and solve for ϕ as a function $\tilde{\phi}(\tilde{w})$. Differentiating this function, verify that

$$\tilde{\phi}'(\tilde{w})=\{(\pi-\eta)\overline{w}\tilde{\mu}'-\pi[(\tilde{w}\mu-\delta)\tilde{Z}'/(1-Z)^2+1/(1-Z)](\tilde{w}\tilde{\mu}'+\mu)\}/(\Omega\mu)>0, \tag{47}$$

after using (45) and the definition of ϕ; recalling that $\eta>\pi$, $\Omega>0>\pi$ and $\tilde{w}\mu>\delta$; while noting that Z and μ satisfy (42) and (43). Moreover, evaluating $\tilde{\phi}(\tilde{w})$ at the point where $\tilde{w}=\overline{w}$, obtain

$$\tilde{\phi}(\overline{w})=(\delta-\overline{w}\overline{\mu})/[(\overline{w}\overline{\mu}-\delta)\overline{Z}/(1-\overline{Z})+\theta\overline{w}\overline{\mu}]<0, \tag{48}$$

where $\overline{Z}$ and $\overline{\mu}$ are the same as in the laissez-faire case; and $\overline{w}\overline{\mu}=\tilde{w}\mu>\delta$ by (15).

Since $\tilde{\phi}'(\tilde{w})>0>\tilde{\phi}(\overline{w})$ by (47) and (48), clearly $\tilde{w}\overset{<}{\underset{>}{=}}\overline{w}$ as $\phi\overset{<}{\underset{>}{=}}\tilde{\phi}(\overline{w})$. Thus, defining $\Phi\equiv\tilde{\phi}(\overline{w})$, we have the following result.

Proposition 6. The steady-state tax on wage income is greater than, equal to or less than zero as ϕ is respectively less than, equal to or greater than Φ .

Of course, the actual value of ϕ depends on the initial conditions (D_0, K_0 and Z_0), basic parameters (b , q , δ , θ and ρ) and production function (F). To confirm that it is indeed possible to have $\phi<\Phi$—and hence a positive tax on labor in steady state—we provide numerical analysis of this case in the next section.

Proposition 6 clearly departs from Shapiro and Stiglitz (1984), who prescribe instead a take-home wage equal to the average product of labor. In fact, in steady state where $\dot{D}=0$ and $\tilde{r}=r$, (26) implies that $\tilde{w}-F(K,Z)/Z=-r(K+D)/Z<0$, which directly contradicts the Shapiro-Stiglitz prescription. Even in the short run with $\tilde{r}=0$ and $\tilde{w}-F(K,Z)/Z=\dot{D}/Z$, their prescription holds if and only if it is optimal for the government to maintain a balanced budget (with $\dot{D}=0$ for all $t<T$).[13] However, the preferred path of public debt might instead be downward over time, implying a take-home wage below labor's average product, as the next section illustrates numerically.

[13] Under a balanced-budget rule requiring $D=0$ for all t, we could readily establish the following implication for the long run. In steady state, if the interest tax is set at its optimal level (still zero), the wage tax must equal zero.

6. Numerical Analysis

This section undertakes numerical analysis of our model in a discrete-time version at quarterly frequency. After discussing some key features of this version,[14] we present two numerical examples, which shed additional light on the optimal-policy analysis of the preceding section.

The household chooses C_t and S_t to maximize

$$U = \sum_{t=1}^{\infty}[C_t^{1-\theta}/(1-\theta) - \delta Z_{t-1}(1-S_t)]/(1+\rho)^{t-1}, \tag{49}$$

subject to $X_t - X_{t-1} = \tilde{r}_t X_{t-1} + \tilde{w}_t Z_{t-1} - C_t$, $Z_t - Z_{t-1} = a_t(1-Z_{t-1}) - (b+qS_t)Z_{t-1}$ and $S_t(1-S_t) = 0$ (as well as initial conditions). The government chooses $\tilde{r}_t$ and $\tilde{w}_t$ to maximize U in (49) with $S_t = 0$, subject to $K_t - K_{t-1} = Y_t - C_t$, $Z_t - Z_{t-1} = a_t(1-Z_{t-1}) - bZ_{t-1}$, $D_t - D_{t-1} = \tilde{r}_t(D_{t-1}+K_{t-1}) + \tilde{w}_t Z_{t-1} - Y_t$, $\mu_{t+1} - \mu_t = \rho\mu_t - \tilde{r}_{t+1}\mu_{t+1}$ and $\tilde{r}_t \geq 0$; where $C_t = \mu_t^{-1/\theta}$ and $a_t = q\tilde{w}_t\mu_t/\delta - \rho - b - q$; while Y_t denotes aggregate output during period t. Note that X_{t-1}, Z_{t-1}, K_{t-1} and D_{t-1} are measured at the end of period $t-1$. The production function has the Cobb-Douglas form, with $Y_t = AK_{t-1}^{\alpha}Z_{t-1}^{1-\alpha}$.

Given K_0, Z_0 and D_0, the optimal solution to the government's problem determines ϕ, as well as the number of periods in the early interest-tax regime (throughout which $\tilde{r}_t = 0$). In the discrete-time version, we allow $0 \leq \tilde{r}_t \leq r_t$ during one period of transition between the early regime and the final regime (with $\tilde{r}_t = r_t$), as do

[14] Further details about optimality conditions, solution methods and calibration procedures are available from the authors.

Chari and Kehoe (1999) for the full-employment case. Our solution also requires that $\gamma_1 = 0$, because μ_1 is unconstrained.

To construct a numerical example in which the wage tax converges to a specific value in the long run, proceed as follows. First, using the steady-state relations calibrated for a hypothetical economy, determine the ϕ corresponding to the specified tax; that is, find $\phi = \tilde{\phi}[\overline{w}(1-\tau_w)]$, dropping the time subscript when referring to the value of a variable in steady state. For the calibration, assume that $Y = 1$ (by choice of units), $Z = 0.95$ (for an unemployment rate of 0.05 in steady state), $\alpha = 0.4$, $\rho = 0.01$, $\theta = 0.5$, $b = 0.45$ and $q = 0.1$.[15] This calibration determines the values of parameters A and δ. The next step is to derive a set of K_0, Z_0 and D_0 consistent with the above-determined ϕ.[16] In this derivation, we first choose K_0 and Z_0, and then solve the model dynamically to determine D_0.

Table 1 shows two alternative sets of initial conditions that imply a wage tax of 1% in steady state. In the first example, capital and employment are initially 10% below their steady-state levels, while the initial level of debt is 123% of annualized steady-state output ($4Y$). For the second example, K_0 is reduced to 25% of K, while Z_0 is unchanged, and D_0 is reduced by roughly 50%.

[15] These values for α, ρ, b and Z are within the range generally used respectively for the labor share in GDP, the long-run quarterly real rate of interest, the quarterly separation rate in the labor market and the long-run rate of employment. Although there is disagreement about the value of θ and no satisfactory estimate for q, our numerical results are not qualitatively sensitive to these two parameters.

[16] Our method is similar to the procedure suggested by Chalmley (1986) for obtaining a solution in the continuous-time version of his full-employment model. To the best of our knowledge, however, a numerical dynamic solution of such a two-regime model has not been provided before.

We now describe the optimal paths of the interest tax, wage tax and public debt for both sets of initial conditions. The length of the early interest-tax regime is 13 quarters for the first example, and 8 quarters for the second. This regime thus lasts longer in the case where the initial capital stock is higher. For both examples, the wage tax is originally negative but increasing, as Figure 1 illustrates. This tax becomes positive in the last period of the early regime, reaches its maximum value in the next (transitional) period, and then declines gradually towards its steady-state level. As Figure 2 demonstrates, debt is always declining—especially in the early regime, where part of the interest-tax revenue is used for this purpose—thereby allowing a smaller tax on labor in the final regime (where interest is untaxed). The debt reduction in the early regime implies that even in the short run, optimal fiscal policy departs from the Shapiro-Stiglitz (1984) prescription for a take-home wage equal to the average product of labor.[17]

7. Conclusion

Referring to employees in the Shapiro-Stiglitz (1984) model, Blanchard and Fischer (1989, p. 458) observe that "once a shirker, always a shirker", because of the exclusive focus on steady state. This observation does not apply to our model, which deals also with transitional dynamics. Outside steady state, the no-shirking wage is affected by changes in the unemployment rate, via adjustments in the job-acquisition probability. Because we introduce asset accumulation, moreover, this wage is also affected by the shadow price of private wealth. Thus, the propensity to shirk can vary over time, and the equilibrium wage must consequently follow a path that (just barely) extinguishes shirking at each instant.

[17]As discussed in section 5 above, this equality fails to hold whenever debt is changing while interest is maximally taxed.

Since our efficiency-wage model of unemployment is not restricted to steady-state analysis, we can investigate the optimal path of factor-income taxes, as Chamley (1986) does for the standard full-employment case. Our investigation not only reconfirms his results on taxation of capital income, but also challenges the policy prescriptions of Shapiro and Stiglitz (1984). Most notably, we find that employment should be taxed (not subsidized) in the long run, if the labor-market distortion is large enough.

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Table 1. Initial Conditions for a 1% Wage Tax in Steady State

Example	K_0 / K	Z_0 / Z	$D_0 / 4Y$
1	0.90	0.90	1.23
2	0.25	0.90	0.60

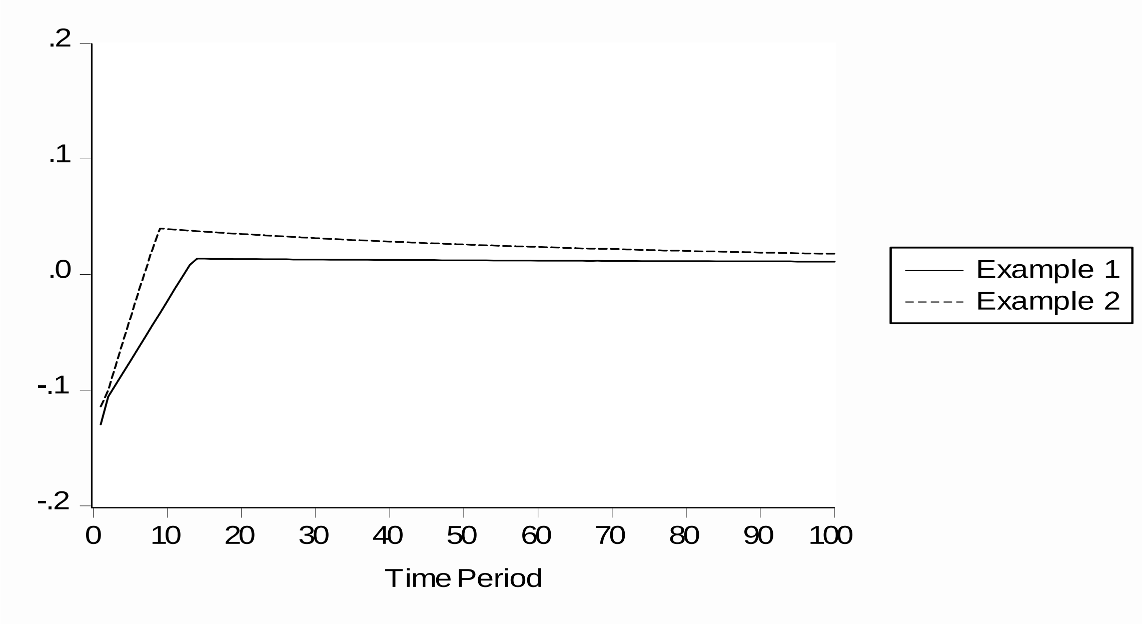

Figure 1. Time Paths of Wage Tax

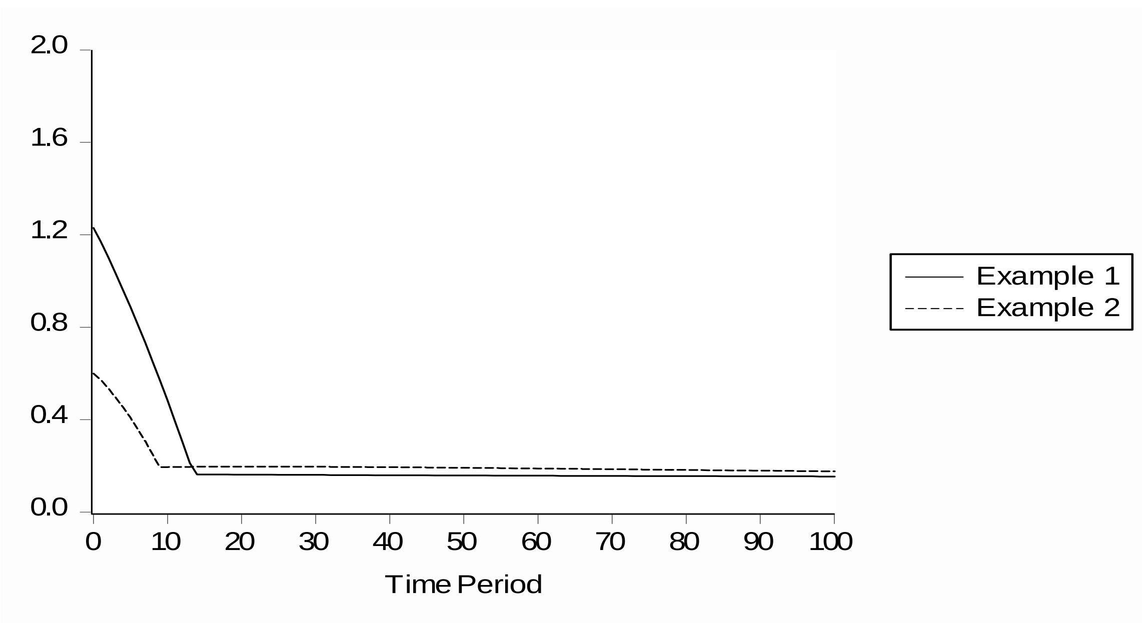

Figure 2. Time Paths of Public Debt